Product Formalisms for Measures, Multiplicative Noise, and Geometric Signal Representation

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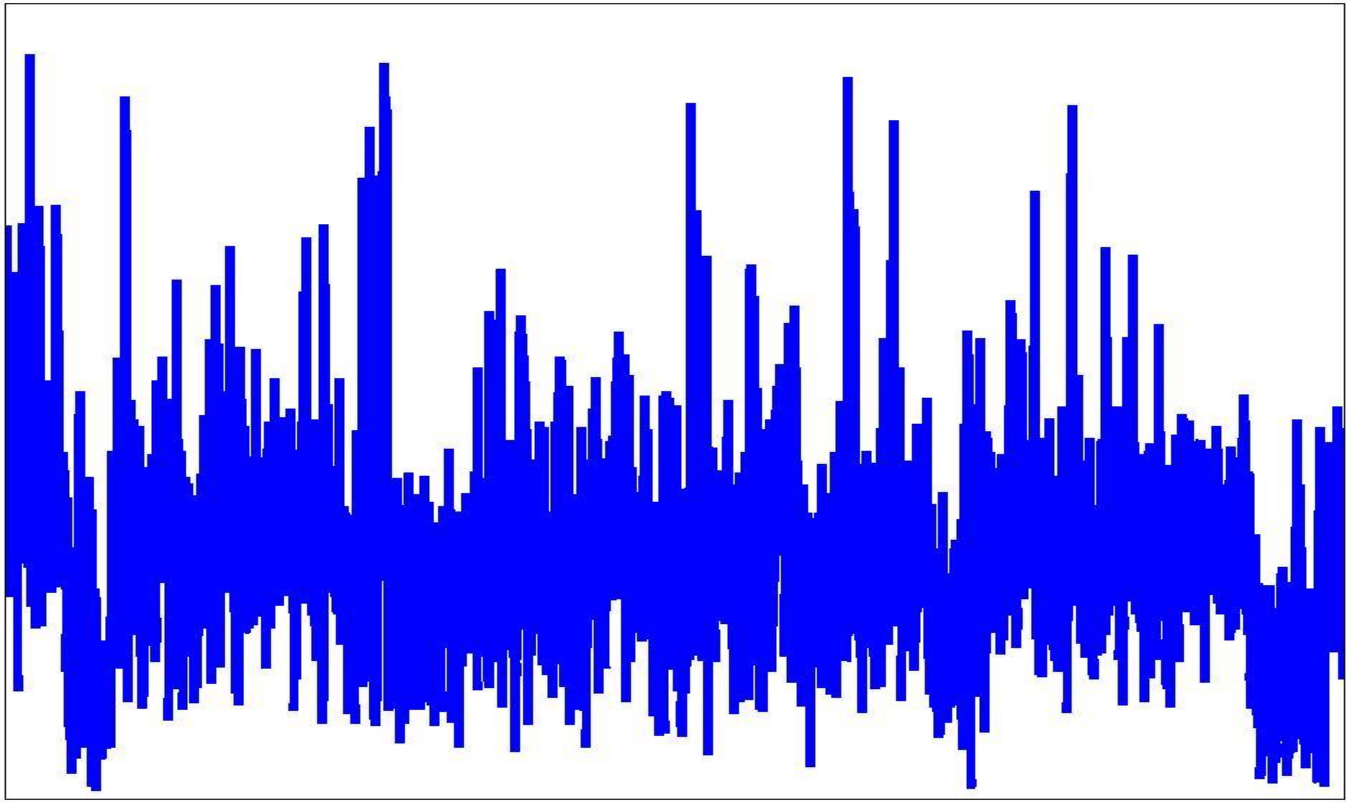
Measures as Data

- What is a measure
- Where do they arise?
- Bursty signals: Internet Traffic, Stock Volume, ……

Bursty Internet Traffic

- http://www.ece.rice.edu/~shri/alphabeta



“SLE(4)”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Volume
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Bad Ideas and Good Ideas

Product Formula Representation Theorem

- Theorem (RF, CK, JP): A non-negative measure μ on the sigma algebra generated by sets in an ordered binary set system $\mathfrak{S}$ on a set X has a unique representation

$$\mu = \prod_{S} (1 + a_S h_S) dy$$

Any assignment of product coefficients in [-1,1] determines a positive measure of total measure = 1 .

Similar formula in any dimension: Any assignment of product coefficients in $[-1,1]^d$ determines a positive measure of total measure = 1 .

- The product formula for measures in the unit interval (for the dyadic sub-intervals binary set system) appeared in "The Theory of Weights and the Dirichlet Problem for Elliptic Equations" by R. Fefferman, C. Kenig, and J.Pipher (Annals of Math., 1991)
- Kolacyzk and Nowak (Annals of Statistics, 2004) also researched multiscale probability models.

Some Haar-like functions on [0,1]

"The Theory of Weights and the Dirichlet Problem for Elliptic Equations" by R. Fefferman, C. Kenig, and J.Pipher (Annals of Math., 1991). We first define the "L^∞ normalized Haar function" h_I for an interval I of form $[j2^{-n},(j+1)2^{-n}]$ to be of form

$$h_I = -1 \text{ on } [j2^{-n},(j+1/2)2^{-n})$$

and

$$h_I = +1 \text{ on } [(j+1/2)2^{-n},(j+1)2^{-n}).$$

The only exception to this rule is if the right hand endpoint of I is 1. Then we define

$$h_I(1) = +1.$$

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Dyadic intervals and +/- 1

Let I denote a (dyadic) interval. Define

$h_I(x) =$ -1 on the left half of I

+1 on the right side of I

0 outside of I

Note: $h_I(x)$ is an L^∞ normalized Haar function.

Probability Measures on [0,1] are given by a canonical product

Any measure arises uniquely as

$$\Pi(1 + a_I h_I(x)) = \mu$$

$$\text{Here } -1 \le a_I \le +1$$

The product is taken over all dyadic intervals, and we order them by the length of the intervals. (The partial products converge in an appropriate topology.)

(Uniqueness: Once the partial product = 0, we define all "following" coefficients to be = 0.)

Why use this representation?

Suppose $F(x) \geq 0$ on [0,1] and

$$\int F_n(x)dx = 1.$$

Then $\log(F(x))$ could $= -\infty$ on a very large set (e.g. on [0,1/2]). Therefore the $\log(F(x))$ could be a very bad function to study. In statistical physics this happens "all the time".

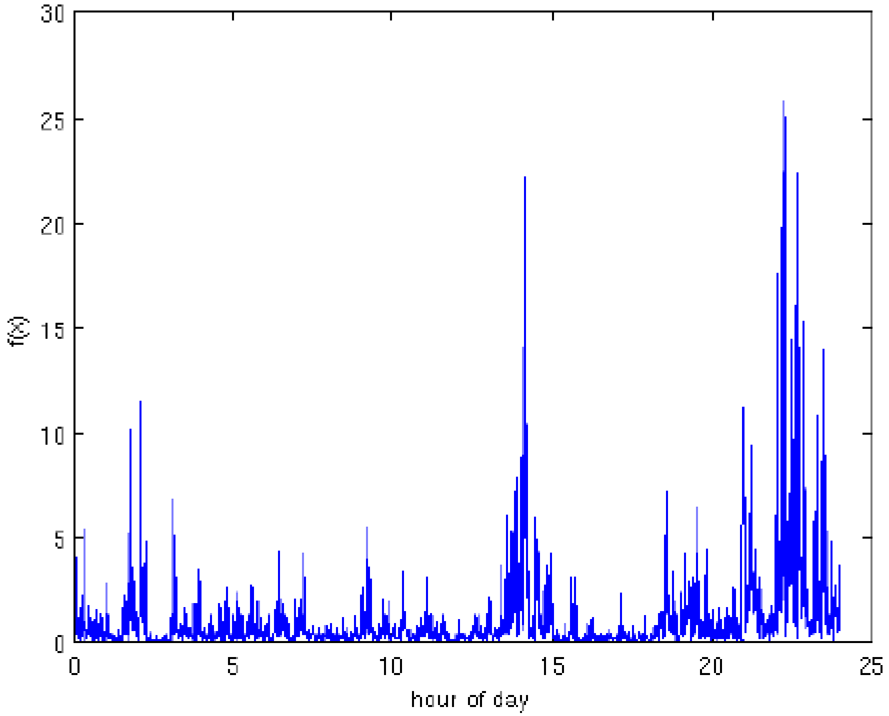


Figure 1 Volume for PDF 1/2(1 + cos(πx))

This is a simulated measure with coefficients chosen randomly from a particular “PDF”.

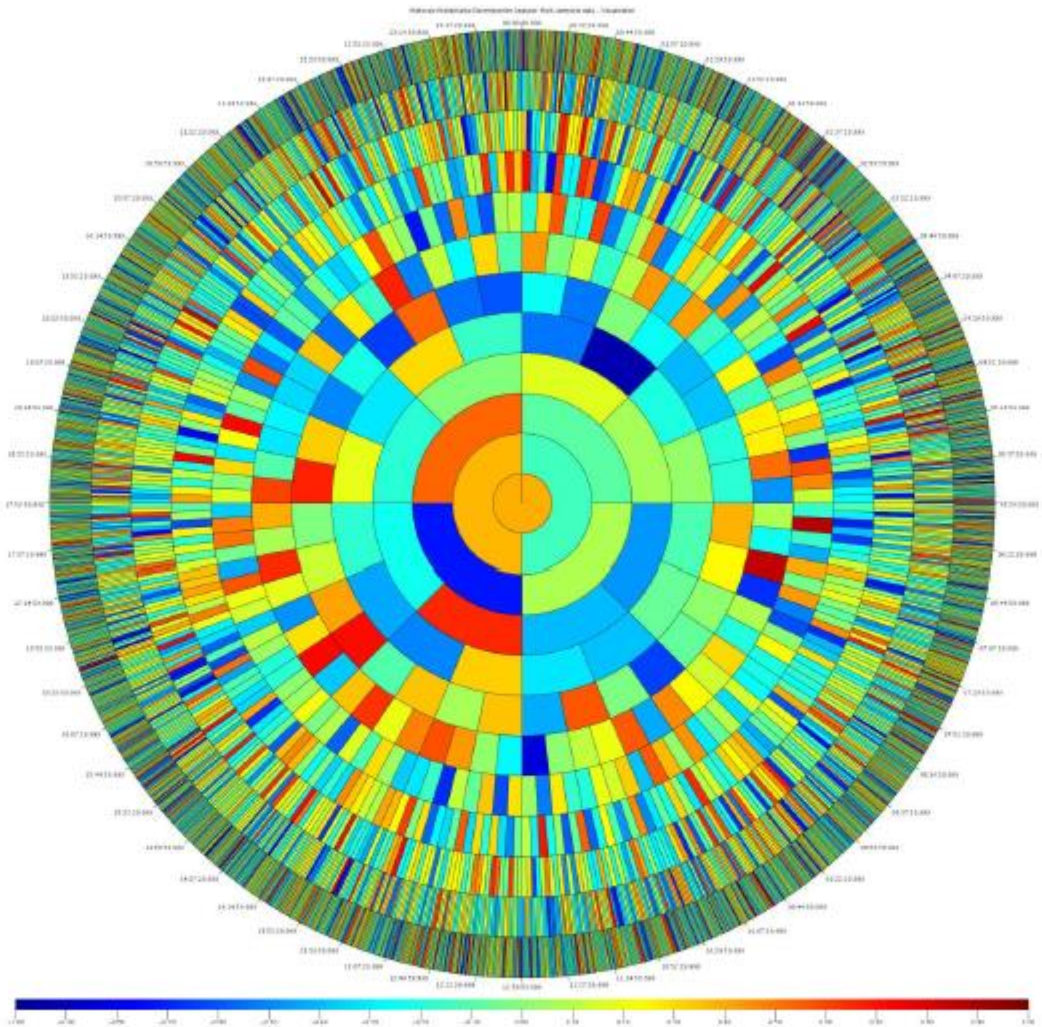
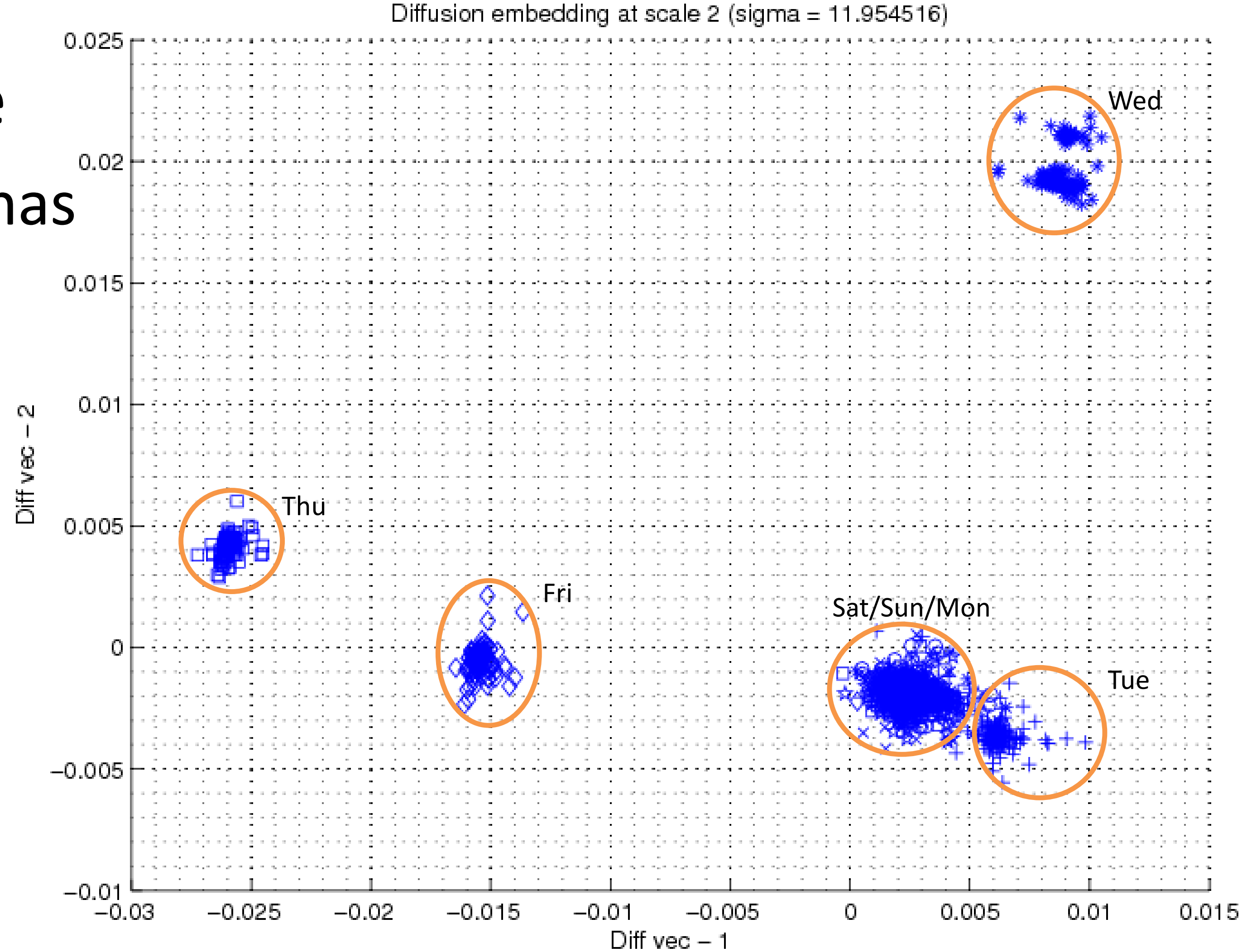


Figure 3 Coefficients for PDF 1/2(1 + cos(πx))

Cell Phone Dataset : PDPM + Diffusion Map

- Daily profile
  - 182 antennas
  - 14 days



Volume from various antennae have had coefficients extracted, embedded by DG.

The Gaussian Free Field

The Gaussian Free Field is an everywhere Divergent random sum which has “the same energy at every scale”. In all dimensions it can be defined by Fourier Series. (I will only discuss [0,1].)

Surprisingly, a theorem due to J.P. Kahane states that if the “variance at each scale” is < 2, one can subtract infinity, exponentiate it, and get non-zero, finite measures (a.s.).

Brownian Motion and the GFF (~!)

1. Brownian Motion: t = time ≤ 1.

$B(t) \sim a_0 t + \sum (a_n/n)\cos(\pi n t)$ (+ sines also)

Notice the "harmless" linear term.

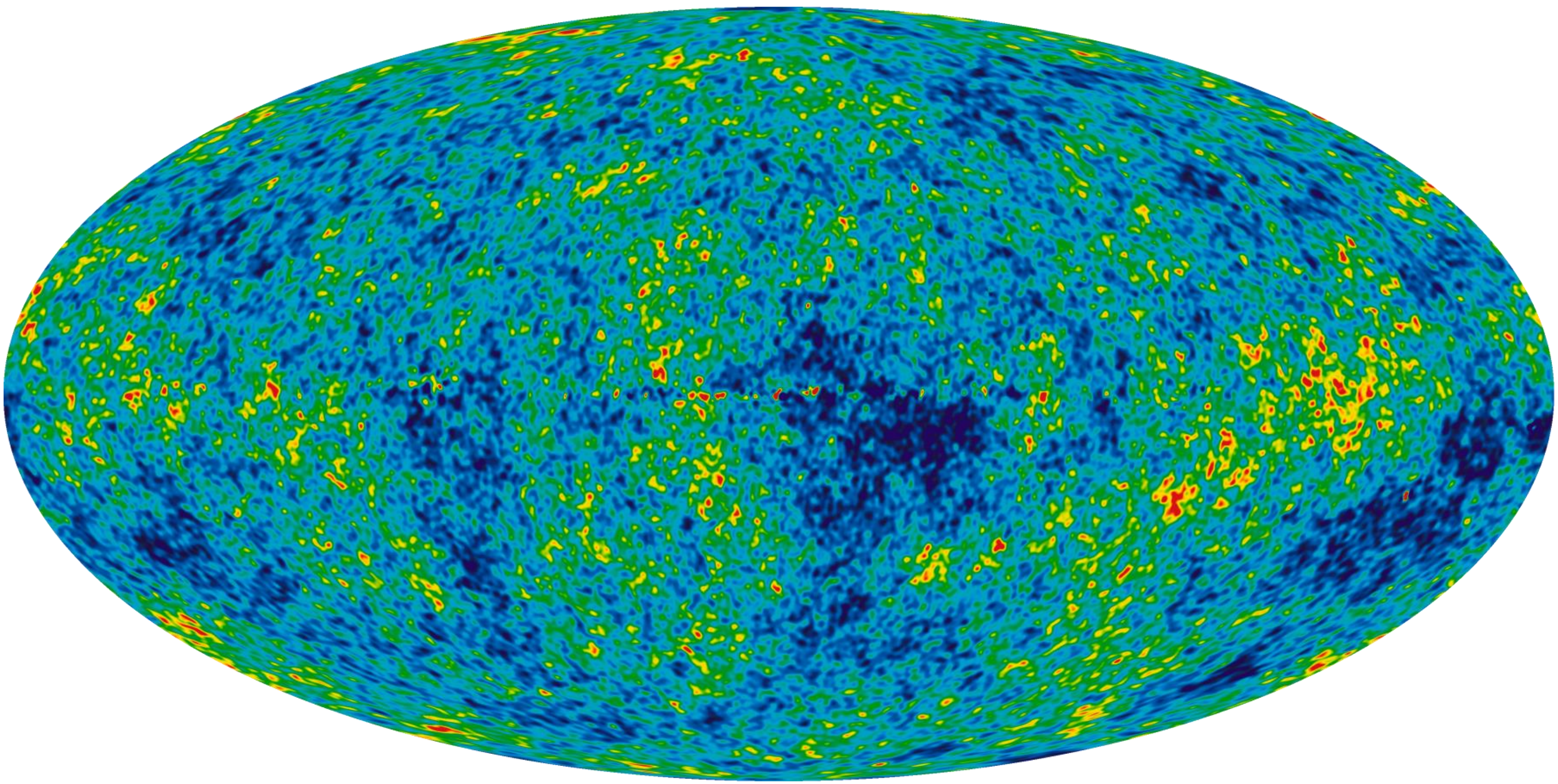
(The sum is INFINITE. And each a_n is a “random Gaussian”, sampled from the bell curve.)

2. Restriction of 2D Gaussian Free Field on S^1, :

$GFF(e^{ix}) \sim \sum (a_n/n^{1/2})\cos(\pi n x)$ (+ sines also)

Notice THE POWER 1/2 (and no linear term).

GFF in Physics: The Cosmic Microwave Background: Deviation From The Mean (See "WMAP")



Kahane:

$\exp\{\sum ((a_n/n^{1/2})\cos(\pi nx)$ (+ sines also) $- \sigma^2/2n)\}$

gives (a.s.) a nonzero, finite measure if $\sigma^2 < 2$.

Dyadic Version:

$\exp\{\sum a_I\, h_I(x)) - (\sigma^2/2)\, \chi_I(x)\}$ has the same properties if

$\sigma^2 < 2\log(2)$. (Note the sum is highly nonconvergent!)

Def. Call F_n the function obtained by summing all terms to scale 2^{-n}.

Review Article on Kahane’s Work

http://www.researchgate.net/publication/236936153_Gaussian_multiplicative_chaos_and_applications_A_review

Theorem on multiplicative noise

Suppose μ is given by a product formula where $|a_n| \leq 1 - \varepsilon$. Then if $\sigma^2 < \varepsilon/2$, a.s. there is a limit measure (from $F_n(x)d\mu$), and

$$\mathbf{0 < \lim} \int F_n(x)d\mu(x) \mathbf{< +}\infty$$

Here

Note: The bound $\varepsilon/2$ can be improved to a more complicated form that is "almost" ε, but there is a constant C so that if $\sigma^2 > C\,\varepsilon$, then almost surely, the limit of $F_n(x)d\mu$ is the zero measure.

Kahane's Theorem and A Multiplicative Noise Model

Let $GFF(\sigma^2)$ denote the Gaussian Free Field with Variance σ^2 and let $GFF_n(\sigma^2)$ denote the partial sum of that GFF with n scales. Define

$F_n(x) = \exp\{GFF_n(\sigma^2)(x) - \tfrac{1}{2} n\sigma^2\}$

(Kahane) Then if $\sigma^2 < 2$, almost surely

$$\mathbf{0 < \lim} \int F_n(x)dx \mathbf{< +}\infty$$

Exists, where the integral is from 0 to 1.

Theorem on multiplicative noise (PJ)

Suppose μ is given by a product formula where $|a_n| \leq 1 - \varepsilon$. Then if $\sigma^2 < \varepsilon/2$, a.s. there is a limit measure (from $F_n(x)d\mu$), and

$$0 < \lim \int F_n(x)d\mu(x) < +\infty$$

Note1: The bound $\varepsilon/2$ can be improved to a more complicated form that is “almost” ε, but there is a constant C so that if $\sigma^2 > C\,\varepsilon$, then almost surely, the limit of $F_n(x)d\mu$ is the zero measure.

Note 2: The Theorem is much more general and states (very approximately) that the measure have a positive measure piece that “behaves like a Cantor measure on a Cantor Set of dimension > 0”.

Conformal Welding

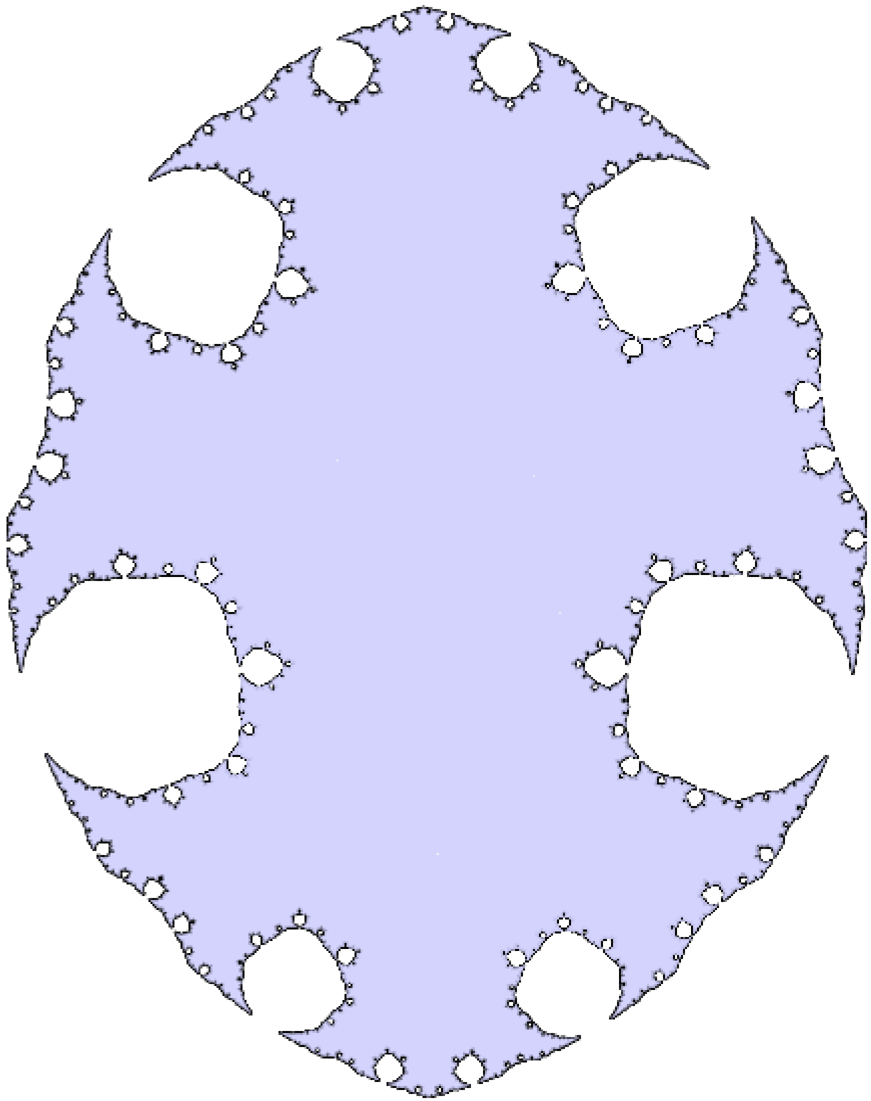
Another tool in QC mapping, arises in simultaneous uniformization.

Let D be a s.c. domain with Jordan curve Γ as boundary. Let F = Riemann map from $\{|z| < 1\}$ to inside domain, G from $\{|z| > 1\}$ to outside domain. Get homeomorphism

$$\Phi = G^{-1} \circ F : S^1 \to S^1$$

There is a well understood mechanism to relate weldings to Quasi-Fuchsian groups and their limit sets.

A Limit Set Produced From A Special "Welding Map". (Bers Theorem)

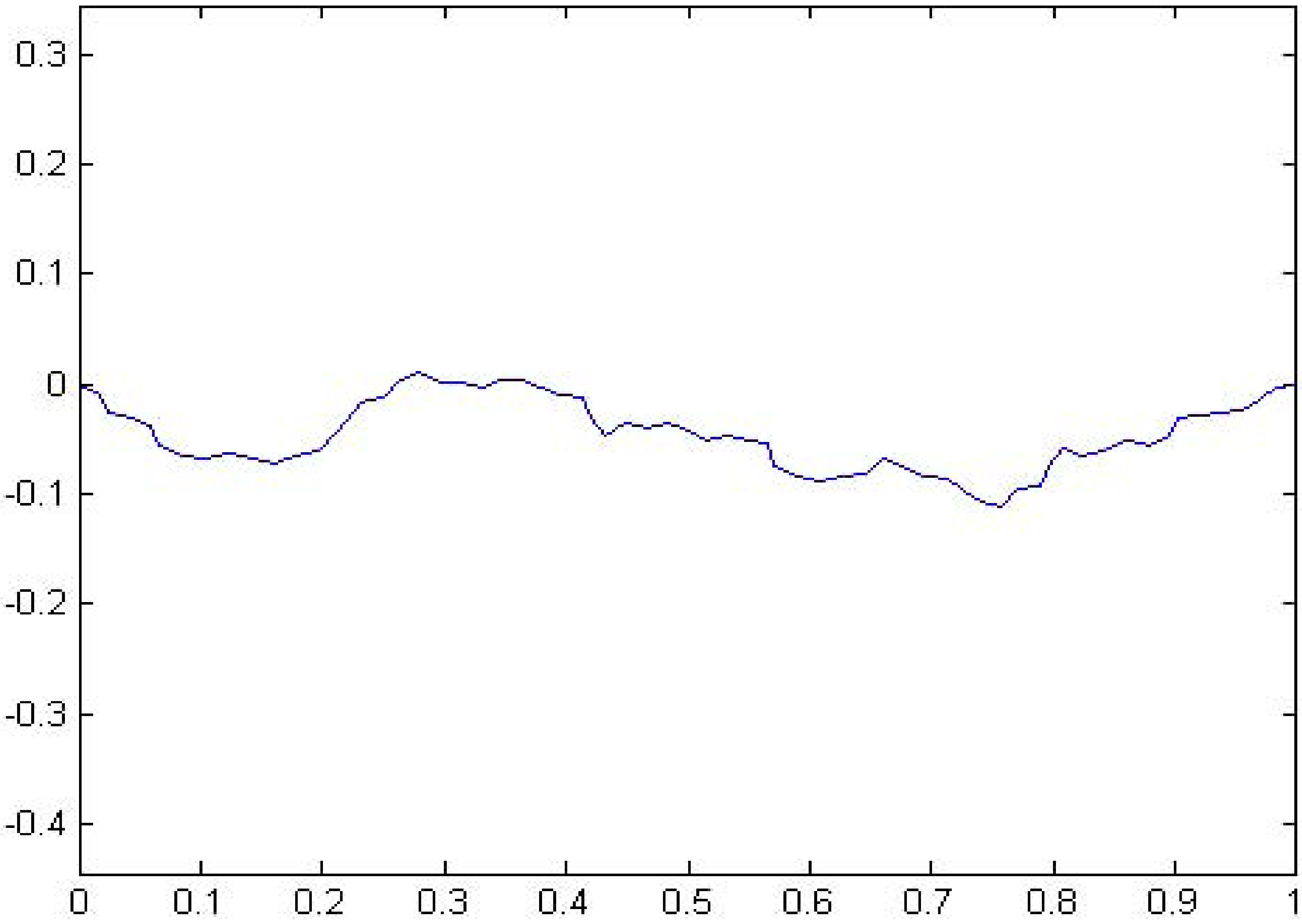


Conditions on a Homeomorphism of S^1 that guarantee a "unique" welding curve

(Ahlfors – Bers Reinterpreted) Let H be a homeomorphism (say increasing) on S^1, and let μ be the derivative of H. Then H is a welding map if there is ε > 0 such that for all rotations of the "dyadic grid" on S^1, all coefficients satisfy

$$| a_I | \leq 1 - \varepsilon.$$

We now show by pictures how to encode a measure as a "pseudo-welding" curve.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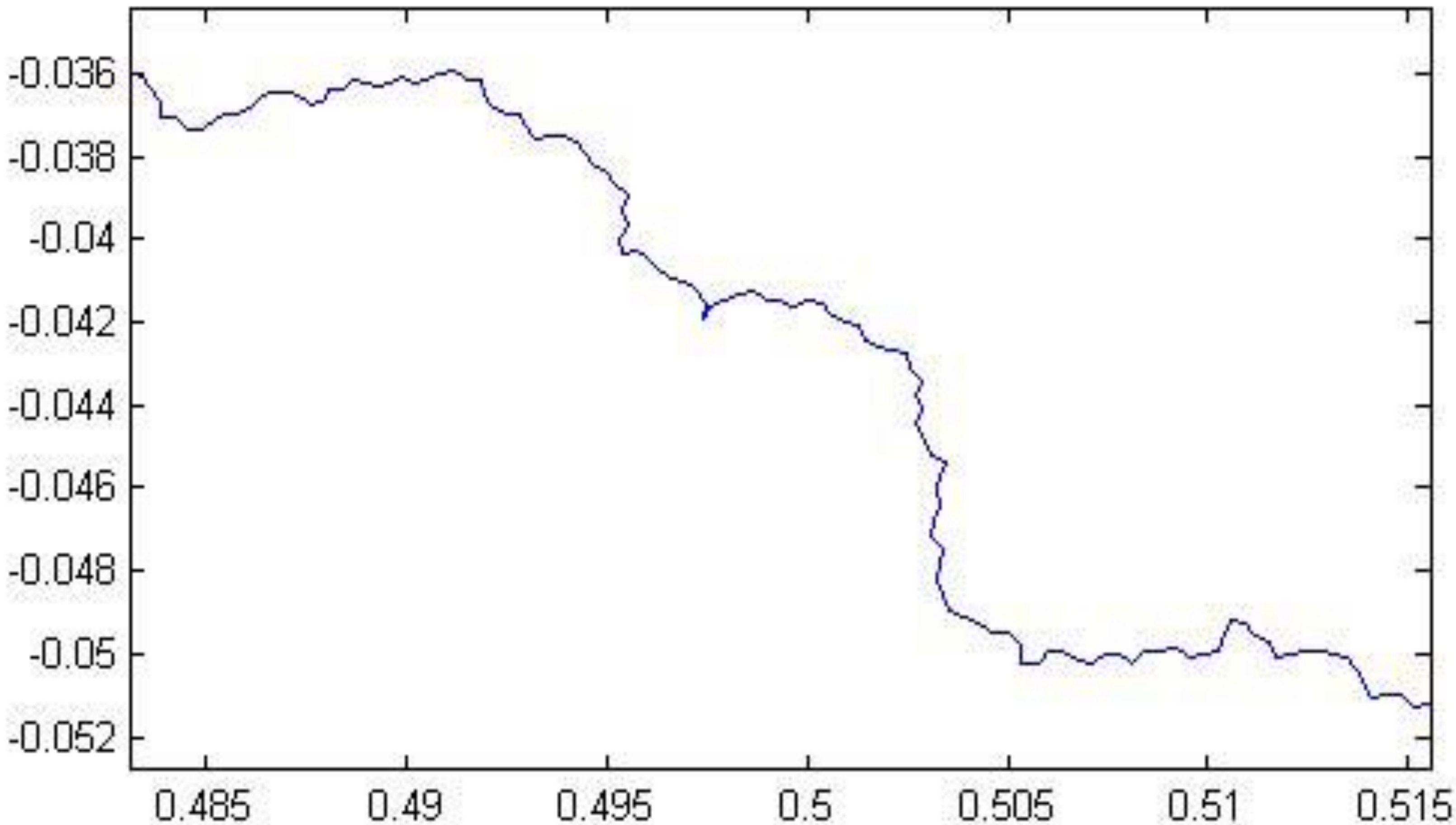
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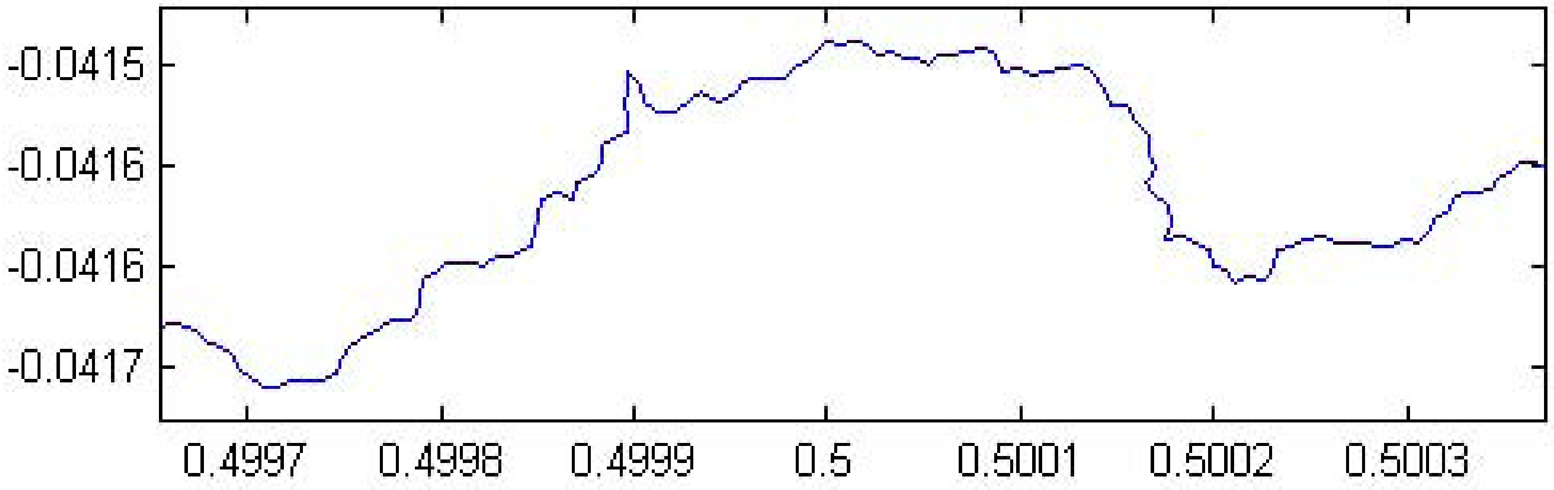
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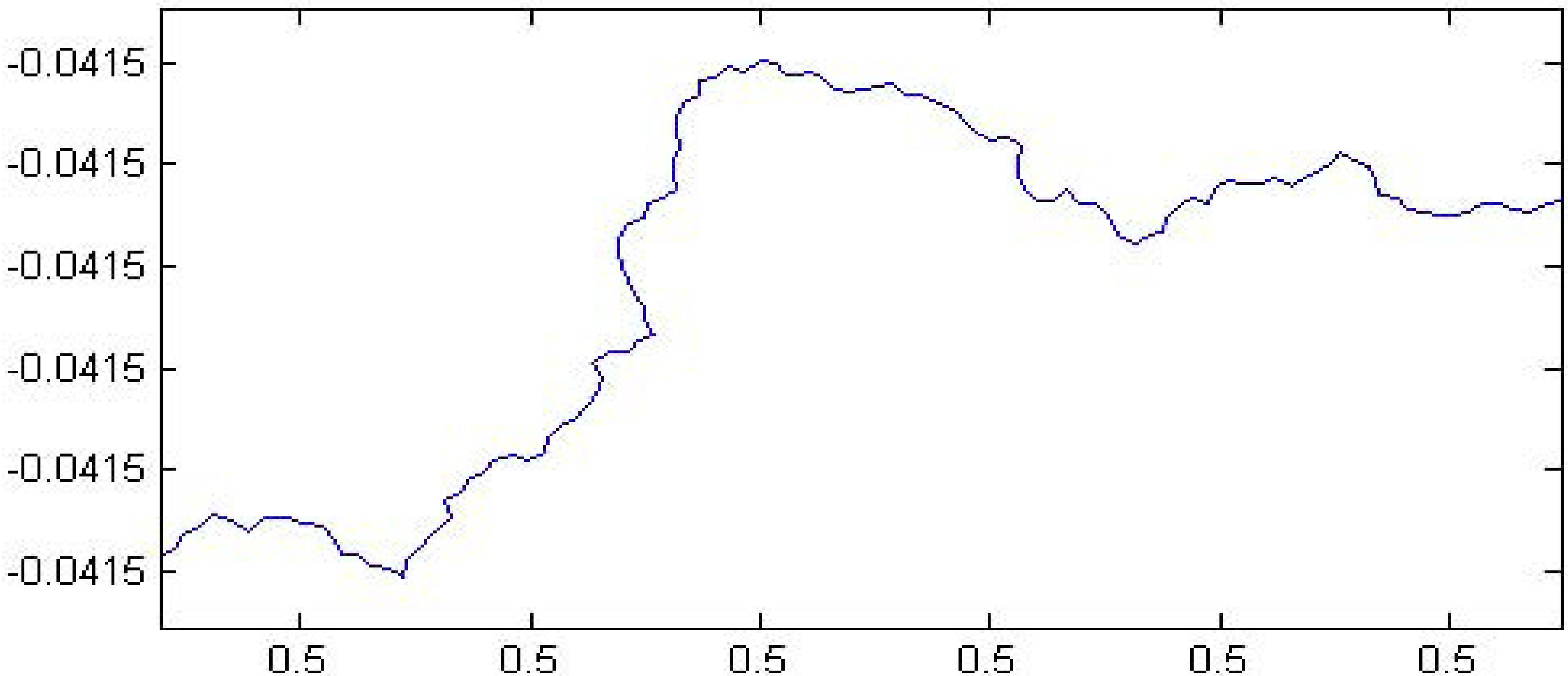

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